Covering Sets for Rectangles in the Lattice: A Variation on a Classic Combinatorial Problem

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Abstract:

This research investigates the minimal covering density for rectangles in the lattice; that is, the smallest possible density of a set of points including at least one corner of every rectangle of a particular size in the lattice ($\mathbb{Z} \times \mathbb{Z}$). After determining the covering density for general $a \times b$ rectangles is $1/4$, we consider pairs of rectangle sizes. We see the density for both $a \times b$ and $b \times a$ rectangles is the same as for just $a \times b$ rectangles. We also prove the minimal density for 1×1 and 2×2 squares is $1/3$. A lemma regarding the integers enables us to find the covering density for $a \times b$ and $a \times d$ rectangles. Finally, the previous result leads to progress on the most general question: determining the covering density for arbitrary $a \times b$ and $c \times d$ rectangle pairs.

1 Introduction: Covering Sets

Of the many major and minor subfields of mathematics, that which is easiest to explain to the non-mathematician is combinatorics, which may be simply described as the science of counting. But as any mathematician knows, the seeming simplicity of this concept conceals a rich variety of challenging problems inspired by and applicable to almost every other area of mathematics. The problem which we will be exploring comes from the subfield of combinatorics known as extremal graph theory. To the mathematician, a graph is simply a set V of vertices and a set of edges $E \subset V \times V$.

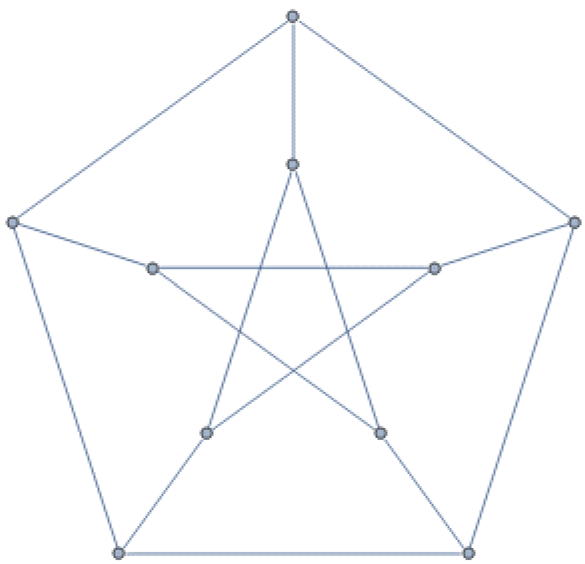

Figure 1: *A mathematical graph.*

Graph theory is the study of these constructs, and extremal graph theory seeks to determine their theoretical boundaries and optimal properties. Perhaps the simplest examples of extremal graph theory are drawn from the family of Turán-type problems.

While Turán-type problems can be applied to many contexts, they are united by the same fundamental goal: maximize a certain quantity without obtaining some undesired configuration. The original Turán-type problem was the question, "What is the maximum number of edges in a graph on n vertices with no K_3?" (Milans & Goldwasser). (As it turns out, the answer is a complete bipartite graph, with the vertices split as evenly as possible between the partitions). A simpler and possibly more famous Turán-type problem is, "Given a finite $n \times n$ grid of points, what is the maximum number of points which can be chosen such that no four points chosen are the four corners of a square?" (*Ibid*). This problem is relatively easy to understand, but has proven highly difficult to answer. For any particular value of n one could program a computer to find the answer, but finding a general formula for arbitrary n is beyond the reach of computation and (thus far) has proven beyond the reach of mathematicians as well, though some bounds do exist (Atjai & Szemeredi).

This latest question leads at last to the research at hand. My research considers a question inspired by the square-free grid problem, but simplified in several key ways. Firstly, rather than considering a finite $n \times n$ grid, we will instead focus on an infinite grid, $\mathbb{Z} \times \mathbb{Z}$, which we refer to as the lattice. While the move to the infinite may at first seem to increase the question's difficulty, much of the complications of the finite problem result from the location of the boundaries, which are eliminated in our case. Our second simplification is that rather than eliminating *all* squares, we will concern ourselves only with avoiding the four corners of some particular size or sizes of rectangle. The output of the question must also be reformatted slightly; in an infinite grid we can always choose infinitely many points, so we will instead ask what fraction of points we may choose from the lattice. Finally, we shall state the problem in the equivalent terms of covering sets; that is, we shall ask what is the minimum density of a set of points chosen from the lattice so that every rectangle of a specified size has at least one corner chosen.

Notation 1.1. In formal terms, we will investigate the function $c : \mathcal{P}(\mathcal{P}(\mathbb{R})) \to \mathbb{R}$ defined by

$$c(R) = \min\{k \in \mathbb{R} : \exists \text{ a density } k \text{ covering set for all rectangles in } R\}$$

(The notation $\mathcal{P}(X)$ here represents the power set of the set X.)

In this paper, we will represent the lattice by a grid of 0s and 1s, with each representing a point of the lattice, and the 0s assumed to form the covering set. While we of course can represent only a finite subset of the lattice on paper, our constructions consist of patterns which are assumed to be repeated infinitely throughout the lattice to form the desired covering set. We can now state our problem as follows:

Problem 1.2. Given $a, b \in \mathbb{N}$, what is $c(\{\text{all } a \times b \text{ rectangles}\})$?

Before attacking the general problem, it will be useful to consider the special case of 1×1 squares –the smallest rectangle which can be fit into the lattice. This proof also serves as an example of the proof technique which we will use most frequently in this paper; we will provide a construction of a covering set with a certain density (giving an upper bound), then prove that there can be no smaller covering set (providing a lower bound and hence completing the proof).

Lemma 1.3. $c(\{\textit{all } 1 \times 1 \textit{ squares}\}) = 1/4$.

Proof. To prove an upper bound on $c(\{\text{all } 1 \times 1 \text{ squares}\})$, take for our covering set the set of all points (i, j) such that $2|i$ and $2|j$ (note that we assume the lower left corner to be the point $(1, 1)$:

$$\begin{matrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{matrix}$$

It can be easily verified both visually and by the fact that any 1×1 square must contain a point with strictly even coordinates that this covering set successfully covers all 1×1 squares, and that this covering set has density $1/4$ in the lattice (since there are three 1s for every 0). Thus, since $c(\{\text{all } 1 \times 1 \text{ squares}\})$ is defined as the minimum over all such covering sets, we see that $c(\{\text{all } 1 \times 1 \text{ squares}\}) \leq 1/4$.

To show that this is the minimum possible covering set, we can obtain a trivial lower bound of $1/4$ by partitioning the lattice into sets of four points as follows:

o	o	o	o	o	o
o	o	o	o	o	o
o	o	o	o	o	o
o	o	o	o	o	o
o	o	o	o	o	o
o	o	o	o	o	o

Each of these sets of four points is the four corners of a 1×1 square, and so our covering set must include one point out of every set of four. Hence we see that $c(\{\text{all } 1 \times 1 \text{ squares}\}) \geq 1/4$.

Thus we conclude that $c(\{\text{all } 1 \times 1 \text{ squares}\}) = 1/4$. □

Having dispensed with this preliminary case allows us to find a solution to our general question.

Proposition 1.4. *For all $a, b \in \mathbb{N}$, $c(\{\textit{all } a \times b \textit{ squares}\}) = 1/4$.*

Proof. Notice that to exclude $a \times b$ rectangles, we can partition the lattice into ab sublattices, where each sublattice consists of all points (i, j) such that $i \equiv x \pmod{b}$ and $j \equiv y \pmod{a}$. One such sublattice exists for each $0 \leq x < b$ and $0 \leq y < a$ (where $x, y \in \mathbb{N}$). For example, to exclude 2×3 rectangles we divide the lattice into 6 sublattices as follows:

4	5	6	4	5	6
1	2	3	1	2	3
4	5	6	4	5	6
1	2	3	1	2	3

From the definition of these sublattices, it follows that any $a \times b$ rectangle has all four corners in the same sublattice, and hence our problem becomes equivalent to excluding 1×1 squares on those sublattices. By Lemma 1.3, it follows that our minimal covering set consists of $1/4$ of each sublattice, and so we see that $c(\{\text{all } a \times b \text{ squares}\}) = 1/4$, as desired. □

This result holds for any rectangles which are parallel to the axes. A natural question, albeit one which marks a more significant departure from the finite-grid problem which inspired our investigations, is whether this result can be extended to rectangles which are not parallel to the axis –for example, the $\sqrt{2} \times \sqrt{2}$ square formed by the four 1s below:

$$\begin{matrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{matrix}$$

As it turns out, it is fairly straightforward to show that the result does extend to this case.

Proposition 1.5. *For any non-axis-aligned rectangle size r, $c(\{\text{all } r\}) = 1/4$.*

Proof. Let r be the rectangle class which we are forbidding, and let v be a point in the lattice. We see that v is at the corner of four rectangles of size r. Each point in those rectangles is also at the corner of four rectangles of size r, and so on. By taking each of these points, we can define a non-axis-aligned sublattice. It is easy to see that we can thus partition the lattice into disjoint sublattices, so that every rectangle of size r is contained in only one sublattice. The example for $\sqrt{2} \times \sqrt{2}$ squares is shown below:

$$\begin{matrix} 2 & 1 & 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 & 1 & 2 \end{matrix}$$

Thus, as before, our problem becomes equivalent to excluding 1×1 squares on these sublattices, and so, by Lemma 1.3, we see that $c(\{\text{all } r\}) = 1/4$. □

2 Multiple Rectangles

Having now completed the discussion of covering sets for single rectangle sizes, it seems natural to extend the question to excluding two rectangle sizes simultaneously. As one would imagine, this question is far more complex and can be broken down into many cases depending on the various divisibilities of the dimensions. Since we were able to determine covering density for both axis-aligned and non-aligned rectangles, it seems logical to ask whether one can cover both. We begin our consideration of multiple-rectangle problems by choosing the simplest example of each: the 1×1 axis-aligned square and the non-aligned $\sqrt{2} \times \sqrt{2}$ square.

Proposition 2.1. $c(\{$*all* 1×1 *and* $\sqrt{2} \times \sqrt{2}$ *squares*$\}) = 1/3$.

Proof. To attain an upper bound, consider the following construction, which has density $1/3$:

$$\begin{array}{cccccccccccc}
1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1
\end{array}$$

It is easy to see that this pattern succeeds in covering all 1×1 and $\sqrt{2} \times \sqrt{2}$ squares with a density of $1/3$, and so we see that $c(\{\text{all } 1 \times 1 \text{ and } \sqrt{2} \times \sqrt{2} \text{ squares}\}) \leq 1/3$.

Now, to prove a lower bound, take any finite $m \times n$ portion of the lattice (a temporary return to the finite grid). Suppose C is a covering set for 1×1 and $\sqrt{2} \times \sqrt{2}$ squares on this portion, and define $q = |C|$. Furthermore, for each $i \in \{1, 2, 3, 4\}$, define S_i to be the number of 1×1 squares which have i corners in C. Since the number of 1×1 squares in an $m \times n$ grid is $(m-1)(n-1)$, we see that

$$S_1 + S_2 + S_3 + S_4 = (m-1)(n-1).$$

Furthermore, since each point in C is at the corner of four 1×1 squares, and each square in S_i contains i points of C, we see that

$$S_1 + 2S_2 + 3S_3 + 4S_4 = 4q.$$

Finally, we know that every 1×1 square has at least one vertex in C. Each point v in C covers four separate 1×1 squares, but, to avoid $\sqrt{2} \times \sqrt{2}$ squares, it must be the case that one of the four points adjacent to v is also in C, otherwise the four points around v would form a $\sqrt{2} \times \sqrt{2}$ square:

$$\begin{matrix} 0 & 1 & 0 \\ 1 & v & 1 \\ 0 & 1 & 0 \end{matrix}$$

This other point will be another corner of two of the four squares which contain v. Thus, since every 1×1 square must have a corner in C, and every point in C must be adjacent to another point in C, we see that for each point in v there can be only two squares in S_1. Hence

$$S_1 \leq 2q.$$

From our first equation, we can see that

$$S_2 + S_3 + S_4 = (m-1)(n-1) - S_1 \geq (m-1)(n-1) - 2q,$$

implying

$$
\begin{aligned}
S_1 + 2S_2 + 3S_3 + 4S_4 &= 4q \\
(S_1 + S_2 + S_3 + S_4) + S_2 + 2S_3 + 3S_4 &= 4q \\
(m-1)(n-1) + S_2 + 2S_3 + 3S_4 &= 4q,
\end{aligned}
$$

and so

$$
\begin{aligned}
4q &= (m-1)(n-1) + S_2 + 2S_3 + 3S_4 \\
&= (m-1)(n-1) + S_3 + 2S_4 + (S_2 + S_3 + S_4) \\
&\geq (m-1)(n-1) + S_3 + S_4 + (m-1)(n-1) - 2q \\
&\geq 2(m-1)(n-1) - 2q.
\end{aligned}
$$

It follows that

$$6q \geq 2(m-1)(n-1),$$

and so

$$q \geq \frac{1}{3}(m-1)(n-1).$$

This proves that the minimum density of a covering set is $1/3$ minus linear terms, which are unimportant as they drop out when we move back to the infinite case. Hence we see that $c(\{\text{all } 1 \times 1 \text{ and } \sqrt{2} \times \sqrt{2} \text{ squares}\}) \geq 1/3$.

Thus we conclude that $c(\{\text{all } 1 \times 1 \text{ and } \sqrt{2} \times \sqrt{2} \text{ squares}\}) = 1/3$. $\square$

During our attempts to prove a lower bound, we actually discovered a second construction:

$$\begin{matrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{matrix}$$

While this result is definitely intriguing, it turns out to be extremely challenging to generalize this to any other pairing of axis-aligned and non-aligned rectangles. As a result, we shall focus the remainder of our covering set investigations on pairs of axis-aligned rectangles. It is this question that has served as the primary focus of our research, and the ultimate motive of the present work is to make progress towards a general solution.

Problem 2.2. What is $c(\{\text{all } a \times b \text{ and } c \times d \text{ rectangles}\})$ for any $a, b, c, d \in \mathbb{N}$?

As one might suspect, this problem has proven extremely unwieldy, and a complete solution is still lacking. However, we can greatly reduce the complexity of the problem by assuming various interdependencies between the dimensions; this approach leads us to some rather interesting results. It is our hope that these special cases shall prove sufficient to eventually inspire a general solution. If one assumes that all four lengths are the same ($a = b = c = d$) then the question reduces to covering 1×1 squares, which, as we have already seen, leads to a covering density of $1/4$. Assuming $a = c$ and $b = d$ reduces the problem to $a \times b$ rectangles, which we have similarly shown to have a covering density of $1/4$. Alternatively, however, one might assume $a = d$ and

$b = c$, in which case our problem becomes to cover both $a \times b$ and $b \times a$ rectangles simultaneously. This seems a natural place to begin our investigations. If one can cover $a \times b$ rectangles with a density of $1/4$, what additional density is necessary to also cover $b \times a$ rectangles? Rather intriguingly, it is actually possible to cover both $a \times b$ and $b \times a$ rectangles with a covering set of just $1/4$, no larger than that for $a \times b$.

Proposition 2.3. $c(\{$*all* $a \times b$ *and* $b \times a$ *rectangles*$\}) = 1/4$.

Proof. Proof of the lower bound follows trivially from the fact that every covering of $a \times b$ and $b \times a$ rectangles is also a covering of $a \times b$ rectangles, and any covering of $a \times b$ rectangles must have density at least $1/4$ by Prop. 1.4. Thus $c(\{\text{all } a \times b \text{ and } b \times a \text{ rectangles}\}) \geq 1/4$.

To prove an upper bound, we will provide construction for three different cases, broken down by the parity of a and b.

If a and b are both odd, then we take for a covering set all points (i, j) such that $2|i$ and $2|j$. Take any $a \times b$ or $b \times a$ rectangle, and let (x, y) be a point on that rectangle. If (x, y) is not in our covering set, then at least one of x, y is odd. If only x is odd, then we know the rectangle contains one of $(x \pm a, y)$ or $(x \pm b, y)$, all of which are in the covering set. Similarly, if y is odd, then we know the rectangle contains one of $(x, y \pm a)$ or $(x, y \pm b)$, all of which are in the covering set. Finally, if both x and y are odd, then we know the rectangle contains one of $(x \pm a, y \pm b)$ or $(x \pm b, y \pm a)$, all of which are in our covering set. Thus we see that this is a valid covering set for $a \times b$ and $b \times a$ rectangles when both dimensions are odd (in fact, this simultaneously covers all rectangles whose dimensions are both odd). Notice that this is the same covering set used for 1×1 squares.

0	1	0	1	0	1	0	1
1	1	1	1	1	1	1	1
0	1	0	1	0	1	0	1
1	1	1	1	1	1	1	1
0	1	0	1	0	1	0	1
1	1	1	1	1	1	1	1

The next case occurs when one dimension is odd and the other is even. Assume without loss of generality that a is odd and b is even. For our covering set, take all points (i, j) such that $i+j \equiv x \pmod{2b}$ for any $x \in \{0, 2, 4, 6, \ldots, b-2\}$ (any even number less than b). Note that this set has density $1/4$. The covering set for 1×4 and 4×1 rectangles is shown below:

1	0	1	0	1	1	1	1
1	1	0	1	0	1	1	1
1	1	1	0	1	0	1	1
1	1	1	1	0	1	0	1
1	1	1	1	1	0	1	0
0	1	1	1	1	1	0	1
1	0	1	1	1	1	1	0
0	1	0	1	1	1	1	1

To verify that this is a covering set, let (x, y) be the bottom left corner of any $a \times b$ or $b \times a$ rectangle. Consider now that if we let $S = \{0, 2, 4, \ldots, b-2\}$, then it is easy to see that S, $S+a, S+b$, and $S+a+b$ are pairwise disjoint sets, with $S \cup S+a \cup S+b \cup S+a+b = \mathbb{Z}_{2b}$. If $x+y \in S$, we are done. If $x+y \in S+a$, then we know either $(x, y+a)$ or $(x+a, y)$ is a corner of our rectangle, and $x+y+a \in S$. If $x+y \in S+b$, then we know either $(x, y+b)$ or $(x+b, y)$ is a corner of our rectangle, and $x+y+b \in S$. If $x+y \in S+a+b$, then we know either $(x+a, y+b)$ or $(x+b, y+a)$ is a corner of our rectangle, and $x+y+a+b \in S$. Thus any rectangle has one corner (x_0, y_0) with $x_0 + y_0 \in S$, and so every rectangle has a corner in our covering set. Another interesting note is that the construction of this covering set does not depend on the value of a, so long as a is odd. Thus the covering set pictured for 4×1 and 1×4 rectangles simultaneously covers 4×3, 3×4, 4×5, 7×4, and so on.

Our last case is when both dimensions are even. Notice that whenever both dimensions share a common factor k, it is possible to divide the lattice into k^2 sublattices so that every $a \times b$ or $b \times a$ rectangle has all four corners in the same sublattice. This is the same process that allowed us to partition the plane into sublattices for a $k \times k$ rectangle in the proof of Prop. 1.4. Thus, when the both dimensions are even, we can partition the lattice into four sublattices:

3	4	3	4	3	4
1	2	1	2	1	2
3	4	3	4	3	4
1	2	1	2	1	2
3	4	3	4	3	4
1	2	1	2	1	2

When we divide the lattice into these sublattices, our problem becomes equivalent to excluding $a/k \times b/k$ and $b/k \times a/k$ rectangles on these sublattices. Thus, whenever a and b are even, by using these sublattices to divide out the gcd of a and b, we can reduce the problem to one of the first two cases (odd by odd or even by even).

Thus we see that for all $a, b \in \mathbb{N}$ there exists a density $1/4$ covering set for $a \times b$ and $b \times a$. Making use of our trivial lower bound, we therefore conclude that
$c(\{\text{all } a \times b \text{ and } b \times a \text{ rectangles}\}) = 1/4.$ □

This proof raises two interesting observations. The first is that if at least one dimension is odd, but the two are not coprime, the proof provides multiple constructions: the odd by odd/odd by even construction, or that provided by first dividing out the common factor. A potentially interesting question for future work is whether these covering sets are the same under any circumstances. A second note is that the constructions provided actually cover far more than we required them to. The odd by even construction simultaneously handles all odd by even rectangles which share the even sidelength, and the odd by odd construction exceeds even that impressive standard by handling all possible odd by odd rectangles simultaneously. This last observation is of sufficient value to warrant a separate statement (though the proof is rather immediate, and is thus omitted).

Corollary 2.4. $c(\{\textit{all odd} \times \textit{odd rectangles}\}) = 1/4$.

Having dispensed with the $a \times b$ and $b \times a$ case, we now turn our attention to the other case which features just two distinct dimensions: $a \times a$ and $c \times c$ squares (that is, taking $a = b$ and $c = d$ in the original problem statement. As with the previous problem, we can immediately dispose of several possible cases. If both a and c are odd, the previous corollary gives us a covering of density $1/4$. If instead both are even (or share any common factors), we may eliminate the shared divisors by partitioning the lattice into sublattices, and thus assume a and c have opposite parity (and, if convenient, no common factors). Unfortunately, due to the relative lack of interdependency between the squares, we have been unable to provide a complete result on this question. As we shall see later, we can provide a construction (and hence an upper bound) for this problem through a more general result, but this construction is likely to be suboptimal in this particular case. The only complete result which we have regarding pairs of squares come from the simplest possible case: excluding 1×1 and 2×2 squares. This is the only result we shall mention in this paper which actually precedes the present research; it served as the original invitation which inspired me to pursue the question further (though it was initially framed in a finite context). The proof presented here is due to Dr. John Goldwasser of West Virginia University.

Proposition 2.5. $c(\{\textit{all } 1 \times 1 \textit{ and } 2 \times 2 \textit{ squares}\}) = 1/3$.

Proof. To show the upper bound, take for a covering the set of all points (i, j) such that $i + j \equiv 0 \pmod{3}$:

1	0	1	1	0	1	1	0	1
1	1	0	1	1	0	1	1	0
0	1	1	0	1	1	0	1	1
1	0	1	1	0	1	1	0	1
1	1	0	1	1	0	1	1	0
0	1	1	0	1	1	0	1	1
1	0	1	1	0	1	1	0	1
1	1	0	1	1	0	1	1	0
0	1	1	0	1	1	0	1	1

For the lower bound, consider a three-row strip of the lattice. To obtain a lower bound of $1/3$, we must show that, on average, every column in this strip has one point in the covering set. Suppose instead we have a column in which none of the three points are in the covering set. If we consider the column to its left, we see that, to avoid 1×1 squares, our covering set must at least contain either the middle point or both the top and bottom point:

1	1	0
0	1	1
1	1	0

Furthermore, if our covering set contains only the middle point in one of the adjacent columns, it must contain one of the outside points in the other adjacent column to cover the 2×2 square shared by the columns to the left and right of our original column (this column must also contain the middle or other outside point to avoid forming a 1×1 square with the middle column). Thus, for every column with no points in the covering set, there is an adjacent column with two points in the covering set. This column with two points in the covering set cannot be adjacent to two columns with no points in our covering set, as those two columns would then have an uncovered 2×2 square. Hence for every column with no points in the covering set there is a unique column with two points in the covering set, and so the overall density of the covering set in our three-row strip is at least $1/3$. Since this holds for every three-row strip of the lattice, we conclude that any covering set for 1×1 and 2×2 squares in the lattice must have density at least $1/3$, and so, since we have a construction of density $1/3$, we conclude that $c(\{\text{all } 1 \times 1 \text{ and } 2 \times 2 \text{ squares}\}) = 1/3$. $\square$

The other pair of squares which we studied is when $a = 3$ and $b = 2$. While we have not been able to prove the exact value in this case, we do know that the value is between $1/4$ and $3/10$ (that is, $1/4 \leq c(\{\text{all } 2 \times 2 \text{ and } 3 \times 3 \text{ squares}\}) \leq 3/10$). The lower bound is the trivial one, and the upper bound is given by the following construction:

$$\begin{array}{cccc} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{array}$$

While we do not have any other useful results for the $a \times a$ and $b \times b$ case, our work has led us to speculate.

Conjecture 2.6. $1/4 \leq c(\{$*all* $a \times a$ *and* $b \times b$ *squares*$\}) \leq 1/3$ *for any* $a, b \in \mathbb{N}$.

The lower bound is the trivial bound; the motivation for this conjectured upper bound will be discussed in greater detail later.

Having considered each of the possible cases featuring just two distinct sidelengths, we now turn our attention to a final simplification featuring three distinct lengths. We have no results specifically regarding $a \times b$ and $c \times c$ rectangles, but they will be at least partially addressed by a more general result discussed below. Assuming $a = c$ while b and d are distinct, we seek to determine a covering set for $a \times b$ and $a \times d$ rectangles (which is of course equivalent to covering $a \times b$ and $c \times b$ rectangles). In order to do so, we first consider a lemma regarding the natural numbers, the proof of which we shall accomplish using some well-known results from graph theory

Lemma 2.7. *If C is a covering set for distances of b or d in the integers (for some $b, d \in \mathbb{N}$), then the density of C in $\mathbb{Z}$ is at least $(\lceil (b+d)/2 \rceil)/(b+d)$, and a covering set with this density exists.*

(Note that such a covering set is defined similarly to the three-dimensional case: a set $S \subset \mathbb{Z}$ such that if $x, y \in \mathbb{Z}$ with $|x - y| = b, d$, then at least one of x or y is in S.)

Proof. First, if b and d have a common factor k, we see that the problem is equivalent to excluding differences b/k and d/k on the disjoint sets $k\mathbb{Z}, k\mathbb{Z} + 1, \ldots, k\mathbb{Z} + (k - 1)$ (where $k\mathbb{Z} = \{\ldots, -2k, -k, 0, k, 2k, 3k, \ldots\}$); this in turn is equivalent to excluding differences of b/k and d/k on $\mathbb{Z}$. Thus we can reduce the problem until b and d have no common factor.

Now consider the integers modulo $(b + d)$. We can define a graph where the vertices are the integers from 0 to $b + d - 1$, with edges between them if and only if their difference is either b or d. An example for $b = 4$ and $d = 5$ is shown below.

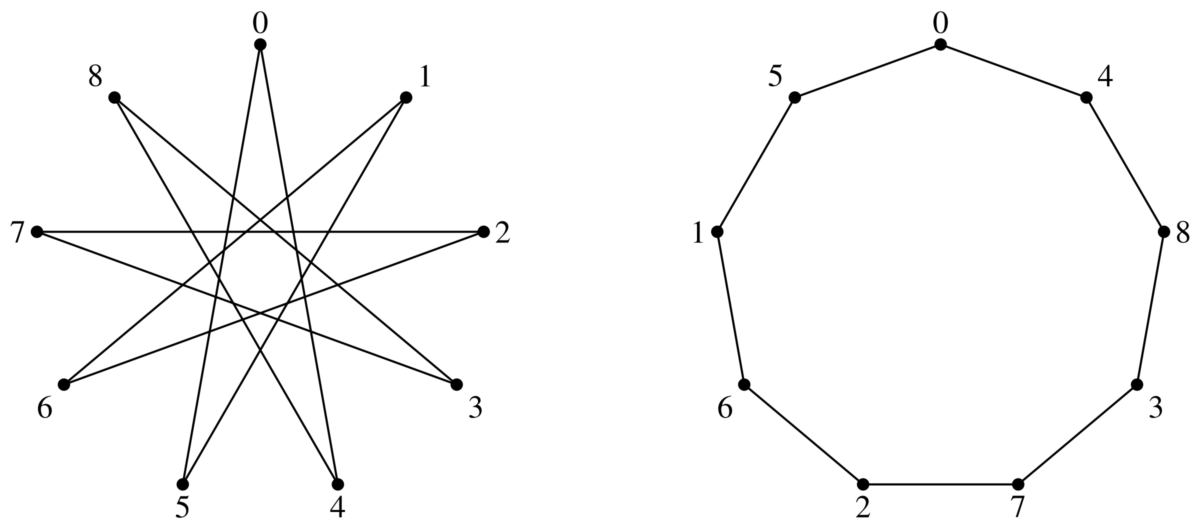

Figure 2: *A graph whose vertices are the integers modulo* 9*, with edges between those whose differences are* 4 *or* 5*.*

Our problem becomes to choose a covering set from these vertices such that every edge has at least one endpoint chosen. But (since mod $(b+d)$ adding b is equivalent to subtracting d and vice versa) we see that each vertex has degree exactly two. It follows that our graph is a cycle, meaning, to choose at least one endpoint of every edge, we must choose at least $\lceil (b+d)/2 \rceil$ vertices. Hence we see that we must choose at least $\lceil (b+d)/2 \rceil$ out of every $(b+d)$ integers, thus obtaining the desired density as a lower bound. However, returning to the cycle, we know we can choose the desired covering set with exactly $\lceil (b+d)/2 \rceil$ vertices, meaning that by repeating this pattern we can obtain a covering set of exactly that density. This completes the proof. □

It should be noted that if $b+d$ is even (meaning both are odd), this covering density simplifies to $1/2$, and the covering set consists of simply choosing every other vertex. This lemma leads to our result on $a \times b$ and $a \times d$ rectangles:

Proposition 2.8. $c(\{$*all* $a \times b$ *and* $a \times d$ *rectangles*$\}) = \lceil (b+d)/2 \rceil / 2(b+d)$.

Proof. Suppose we have a covering set S on the lattice. Partition the lattice into pairs of rows, with each pair distance a apart from each other. For any particular pair of rows (with y-coordinates y_1

and $y_1 + a$) consider the covering set R on the integers defined by

$$R = \{x | (x, y_1) \in S \text{ or } (x, y_1 + a) \in S\}.$$

That is to say, if the part of S which resides in each row may be thought of as a partial covering set on the integers, R is the union of these two partial covering sets. It can be seen clearly that, if S is a covering set as desired for $a \times b$ and $a \times d$ rectangles, than R must be a covering set for differences of b and d on $\mathbb{Z}$, meaning R has density at least $\lceil (b + d)/2 \rceil / (b + d)$, and so, since R is derived from two rows of S (and our choice of a pair of rows was arbitrary), we see by our lemma that S has a density in the lattice of at least $\lceil (b + d)/2 \rceil / 2(b + d)$. However, by the same lemma was can find a covering set R_0 for differences of b or d on the integers with the above density. Define a covering set S_0 by alternating a rows of unchosen points with a rows, each of which contains R_0. It can then be easily seen that S_0 is a covering set as desired, since any $a \times b$ or $a \times d$ rectangle will have either its top or bottom corners in a row containing R_0, and thus, since those corners are distance b or d apart, one of them will be in S_0. Furthermore, S_0 has the desired density $\lceil (b + d)/2 \rceil / 2(b + d)$. This completes the proof. □

An example for $a = 2, b = 4, d = 5$ is shown below:

0	0	0	0	0	1	1	1	1
0	0	0	0	0	1	1	1	1
1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1
0	0	0	0	0	1	1	1	1
0	0	0	0	0	1	1	1	1
1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1

If a is odd, then one may choose to alternate rows with or without covering sets rather than alternating every a rows. Furthermore, we noted above that if b and d are both odd, then their covering set in the integers consists of alternating points chosen and unchosen, and thus has density $1/2$. When all three are odd, if one directly alternates rows as suggested, the resulting covering set will be the covering set for all odd by odd rectangles discussed above.

What the last proof essentially says is that covering $a \times b$ and $a \times d$ rectangles is equivalent to covering differences of b or d on the integers. We were able to prove that the two problems shared a lower bound (or more accurately, that one was a multiple of the other). Then we were able to provide a construction which followed the simple logic of guaranteeing that the top or bottom side of the rectangle is in a row with points chosen, then guaranteeing that one of the corners on that side is chosen. That same concept can be extended to a more general case, or even to the most general case.

Theorem 2.9. *For all $a, b, c, d \in \mathbb{N}$, there exists a covering set for $a \times b$ and $c \times d$ rectangles with density*

$$\frac{\lceil (a+c)/2 \rceil}{2(a+c)} \frac{\lceil (b+d)/2 \rceil}{2(b+d)}.$$

Proof. Let R_1 be an optimal covering set for differences of a or c on the integers, and let R_2 be an optimal covering set for differences of b or d on the integers (as defined in our lemma). Define our covering set S by

$$S = \{(x, y) | x \in R_1 \text{ and } y \in R_2\}.$$

Since by the lemma R_1 has density $\lceil (a+c)/2 \rceil / 2(a+c)$ and R_2 has density $\lceil (b+d)/2 \rceil / 2(b+d)$, it follows that S has the desired density. To see that S is a covering set, notice that any $a \times b$ or $c \times d$ must have either its top or bottom edge in a row containing points of S (since the y-coordinates of these rows forms a covering set for differences of a and c). But similarly, either the left or right corner of that edge must be a point in our set, since the points of S in the row form a covering set for differences of b and d. It follows that at least one corner of any $a \times b$ or $c \times d$ rectangle is in S, and so S is indeed a covering set. □

It is important to notice that, unlike our previous results, this construction comes with no accompanying lower bound. That is to say, while this provides an algorithm to generate a covering set for any pair of rectangles, it is not necessarily an optimal covering set. Indeed, several of the covering sets provided earlier for special cases are smaller than the covering set which this theorem suggests for the same case. (It does however, reduce to the previously presented construction in the case of an $a \times b$ and $a \times d$ rectangle pairing, as above, or for any odd by odd rectangle.) This general covering set is significant primarily because it allows us to place general bounds on the covering density of arbitrary $a \times b$ and $c \times d$ rectangles.

Corollary 2.10. *For all $a, b, c, d \in \mathbb{N}$,*

$$1/4 \leq c(\{\textit{all } a \times b \textit{ and } c \times d \textit{ rectangles}\}) \leq 2/5.$$

Proof. Consider the expression $\lceil n/2 \rceil / 2n$. For any even n, this reduces to $1/2$. Conversely, for odd n the value is strictly decreasing as n increases (excluding $n = 1$). For $n = 3$, this fraction is $2/3$, and for $n = 5$ it becomes $3/5$. The only possible combination of values for which the covering density is greater than $2/5$ is if $(a + c) = (b + d) = 3$. However, this can be true only if $a = b = 1$ and $c = d = 2$ (1×1 and 2×2) or if $a = d = 1$ and $b = c = 2$ (1×2 and 2×1). In either of these cases, we have previously proven that another covering set exists with density less than $2/5$. The next possible value occurs when $(a + c) = 3$ and $(b + d) = 5$ (or vice versa), which results in a covering density of $(2/3)(3/5) = 2/5$. This completes the proof. □

This allows us to finish the discussion of our earlier conjecture that the covering density for any pair of squares is between $1/3$ and $1/4$. In general, the covering set provided here is heavily suboptimal for covering squares or any rectangle pairs with dimensional interdependence. The last corollary serves to prove a weakened form of our conjecture (between $1/4$ and $2/5$), but we have yet to find proof of any case in which the covering density must be greater than $1/3$. Thus, whether the tightest possible upper bound is $1/3$, $2/5$, or something in between remains an open question.

Future Work

As with any mathematical endeavor, the results we have obtained serve only to open further questions for examination. There are several in particular which I find to be promising. Perhaps the most obvious is to determine the optimal covering density for $a \times a$ and $b \times b$ squares, or $a \times b$ and $c \times c$ rectangles. Furthermore, the corollary to our general covering set theorem notes that we can now guarantee a covering density between $2/5$ and $1/4$ for any rectangle pair, but there are only a few specific cases with a density as high as $2/5$. By determining more efficient coverings for those particular cases (or proving such to be impossible) one could make progress towards the question of what is the largest optimal covering density. Finally, the square grid, which we have represented here as the lattice, is just one of 11 regular tessellations known as the Archimedean lattices, each home to a distinct array of shapes and patterns. While the more irregular nature of some of the lattices will make it difficult to directly generalize the notion of multiple rectangle sizes, the mere fact that many appear to be a collection of multiple shapes suggests that it might be intriguing to find covering sets for each of those shapes. And at the least, it should be quite possible to find a covering set for side-length 1 and 2 equilateral triangles in the triangular grid, or even side-length 1 triangles and hexagons.

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