A New Metric for Credit Migration Matrices

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Abstract

Predicting credit risk is essential to the study of risk management. One manner of predicting the credit risk of a financial institutions portfolio is through credit migration matrices. These credit migration matrices include the probabilities of transitioning to a credit rating given these firms current credit ratings. Long term analysis can be performed using these migration matrices to create a Markov chain. This technique can be used to compute how many firms are in each possible credit rating after any amount of time from the current state. However, it is not often intuitive which transition matrices are more desirable to the financial institutions than others. This research supplies a metric as its main result which finds the weighted credit rating for a fixed number of firms given a migration matrix which can be used to compare the desirability of multiple matrices.
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Contents

1 Introduction .......................... 6
   1.1 History of the U.S. Credit Rating Agencies ................................... 7
   1.2 Importance of Credit Ratings ....................................................... 7
   1.3 Importance of Predictions of Credit Migration ................................. 9
   1.4 Markov Chains and Application to Credit Migration .......................... 9
      1.4.1 Mathematical Definition ..................................................... 10
      1.4.2 Application to Credit Migration ........................................... 10

2 Metrics for Credit Migration Matrices .............................................. 13
   2.1 Definition of Metrics ............................................................... 14
   2.2 Criteria for Mobility Metrics .................................................... 14
   2.3 Average of the Singular Values of the Mobility Matrix ..................... 15

3 A New Metric for Credit Migration Matrices ...................................... 16
   3.0.1 Mathematical Definition ....................................................... 17

4 Comparison of Metrics .................................................................. 19
   4.1 Method of Comparison ............................................................... 20
      4.1.1 Construction of Matrices and Analysis .................................... 20

5 Analysis of The New Metric ........................................................... 24
   5.1 Method of Analysis ................................................................. 25
      5.1.1 Constructing Matrices 3 x 3 ................................................. 25
5.1.2 Results for 3x3 Transition Matrices ........................................ 27
5.1.3 Constructing Matrices 5 x 5 .................................................. 29
5.1.4 Results for 5 x 5 Transition Matrices ................................. 30

6 Conclusion and Future Work ................................................. 31

7 5x5 Matrices Used in Comparison of Metrics ................................. 34
7.1 Matrices with Strictly Increasing Value of our Metric ...................... 34
7.2 Matrices with Strictly Increasing Value of Jafry’s Mobility Metric ........ 35

8 Analysis of New Metric .......................................................... 36
8.1 3x3 Matrices Used in Analysis ............................................. 36
8.1.1 Transitions from A to A .................................................. 36
8.1.2 Transitions from B to B .................................................. 37
8.1.3 Transitions from C to C .................................................. 38
8.2 5x5 Matrices Used in Analysis ............................................. 39
8.2.1 Transitions from A to A .................................................. 39
8.2.2 Transitions from B to B .................................................. 40
8.2.3 Transitions from BB to BB .............................................. 41
8.2.4 Transitions from C to C .................................................. 42
8.2.5 Transitions from D to D .................................................. 43

9 Analysis of 5x5 Credit Migration Matrices .................................. 44

10 VBA Code Examples .............................................................. 46
1 Introduction

A financial system connects a bank or other lender with a surplus of money with a person with a shortage of money to arrange for an extension of credit. Extending credit can be done through either direct finance, where lenders and borrowers connect in a market such as a bond market, or through indirect finance where a financial intermediary transfers funds from lenders to borrowers [4]. Examples of financial intermediaries include banks, insurance companies, pension funds, and finance companies. In the public bond market, specialist lenders may have the means to determine whether or not borrowers are worthy of the extension of credit. These financial intermediaries often also have the resources available to them to determine the credit-worthiness of a borrower [17]. However, the general public does not have the individual means to evaluate the financial security of a potential borrower when looking to make an investment. This creates a need for credit rating agencies which provide this information to the general public, namely by providing a credit rating (for example AAA, AA+, BB-, etc.) for the borrower. A credit rating assesses the borrower’s overall financial
health to represent the “relative likelihood that a borrower will default on its obligations ” [7].

1.1 History of the U.S. Credit Rating Agencies

In the United States these ratings are issued by three credit rating agencies: Moody’s, Standard & Poor’s, and Fitch [6]. Before their existence, these credit rating agencies were preceded by mercantile credit agencies which provided ratings for merchants. In 1909, the first credit ratings began being assigned to securities by John Moody, namely railroad bonds. Following Moody, Poor’s Publishing Company also began providing ratings in 1916 followed by Standard Statistics Company in 1922 and Fitch Publishing Company in 1924 [6]. In 1941 Poor’s Publishing Company merged with Standard Statistics Company to form Standard & Poor’s. Due to this long established history, by the end of the twentieth century the three U.S. credit rating agencies, which had expanded substantially internationally, were advantageous over foreign credit rating agencies [6]. In present day there is specialization between the three agencies: Moody’s and Standard & Poor’s most often rate corporations, financial institutions, governments, and asset-backed securities. However, Standard & Poor’s also rates insurance companies [13].

1.2 Importance of Credit Ratings

The credit rating of any significant financial entity, namely corporations, insurance companies, and financial institutions, directly impacts that entity’s financial security [7, 13]. These entities often issue bonds as a means of financing operations. Bonds are debt securities issued by these entities which investors purchase and mature after a predetermined amount of time.
The issuer must then repay the face value of the bond, called the principal, plus interest [1]. Issuing bonds is considered a financial risk for the entities. Sebastian Herzog explains in *Optimal Credit Ratings* that “firms take financial risks on purpose, since interest payments are tax exempt as opposed to dividend payments, which are not. Debt financing, therefore, has tax advantages over equity” [9]. Having a poor credit rating makes attracting investors to these bonds difficult and thus makes it difficult for entities to take the appropriate risks that would be most financially beneficial to them. Another reason obtaining a favorable credit rating is important is that it has been shown that the value of a firm’s assets is linked to its credit risk. A firm will go into default when the value of its assets is less than the value of their callable liabilities [5].

Credit ratings also impact the larger economy and national or local governments [7]. Richard Cantor’s paper “Determinants and Impact of Sovereign Credit Ratings” exemplifies the importance of a country’s credit rating, also called a sovereign credit rating, through discussion of the rating’s microeconomic impact: the credit rating of a country’s local government or private business will almost always be lower than the country’s sovereign credit rating. Since credit ratings are often critical to bond issuers in determining the issuer’s success in financing activities, a country’s poor sovereign credit rating can have a negative impact on both the macroeconomic and microeconomic levels [7]. On the macroeconomic level, the country would have trouble obtaining credit in the international lending market; whereas on the microeconomic level, local governments and private businesses could not be financially successful. When there are problems in the economy or financial markets, risk management professionals perform stress testing on banks’ credit portfolios. This stress testing often involves mathematical analysis using matrices to predict credit migration [5].
1.3 Importance of Predictions of Credit Migration

Predicting changes in credit ratings is critical to risk management professionals who often consider credit migration for the entire credit portfolio of a large financial entity [8]. Upper management of financial institutions often desire a certain level of certainty that the probability of the institution incurring a safe amount of losses is equal to that of the safe amount of losses on its capital [14]. Thus, banks make predictions regarding their financial risk to estimate how much capital they need to back their credit portfolios in order to meet federal regulations along with those of the management [8]. Furthermore, the total value of a firm utilizing debt financing can be modeled by the firm’s equity plus the tax benefits from the debt financing, minus present value bankruptcy costs attached to the firm’s credit portfolio. These bankruptcy costs are directly related to probability of the firm defaulting on its debt financing [9]. Mathematical methods, such as Markov chains, can be used to predict changes in credit ratings.

1.4 Markov Chains and Application to Credit Migration

In the three major U.S. credit rating agencies, there are a finite number of possible credit ratings [2, 11, 16]. Therefore credit ratings are a discrete variable in time which define specific classes or states. We can use Markov chains to represent these “dynamic movements with a matrix cross-classifying [of] the states or classes occupied at two points in time ” [15]. The time between the issuance of credit ratings for a financial entity can be represented either discretely or continuously.
1.4.1 Mathematical Definition

A dynamical system is a set of discrete variables whose values change over time [3]. The vector, or state vector, is comprised of the values of these variables at a particular state [3]. Many of these systems are stochastic processes where the states of the values are represented as probabilities since the state of the variable is not known with complete certainty. A stochastic process in which the state vectors can be related at successive time intervals by the equation \( x(k+1) = x(k)P \) is called a Markov chain. Here, \( x(k+1) \) represents the state vector after an interval of time. \( P = [p_{ij}] \), the transition matrix of the system, is a stochastic matrix where \( p_{ij} \) is the probability of the system being in state \( j \) at the discrete time \( k+1 \) given being in state \( i \) at time \( k \) [3]. It follows that \( k \) is in terms of \( \Delta t \). Therefore it follows to find \( x(k) \), we can also compute \( x(k) = x(0)P^k \) [3].

An alternate method of representing \( P \) can be done through finding the eigenvectors of \( P \) [12]. \( P \) can be represented as

\[
P = V \times D \times V^{-1}
\]

Where \( V \) holds the eigenvectors of \( P \) and \( D \) is a matrix with the eigenvalues of \( P \) on the diagonal [12].

1.4.2 Application to Credit Migration

When applying Markov chains to credit migrations, the system’s state vectors represent the number of firms or other entities at any of the possible credit ratings at time \( t \). These state-vectors contain \( n \) possible states for each possible credit rating [3, 10]. The state-vector with \( n \) possible credit ratings at a time \( t \) is represented:
\[ x(t) = \begin{bmatrix} x_1(t) & x_2(t) & \cdots & x_n(t) \end{bmatrix} \]

Where \( x_1(t) \) is the number of firms with the best credit rating,

\( x_2(t) \) is the number of firms with the next best credit rating

\[ \cdots \]

\( x_n(t) \) is the number of firms with the default credit rating

Figure 1: The construction of a state vector.

Furthermore, the credit migration matrix of the system provides the probabilities of an obligor transitioning to a different credit rating or remaining at its current rating given their current rating [10]. Therefore, given \( n \) possible credit ratings, a credit migration matrix will always be a \( n \times n \) matrix with row-wise entries summing to one. The row-wise entries must sum to one since the firm either stays at its current rating or transitions to one of the remaining ratings [10]. Since the row-wise entries must sum to one, the dynamical system created applying Markov chains to credit migration is neutrally stable. This means the successive state vectors \( x(t) \) will eventually stabilize rather than approach zero or infinity. Thus, we are able to find an equilibrium credit migration matrix and corresponding final state vector [10].

For an example, let \( n = 4 \), with credit ratings A, B, C, and D, where D designates a firm in default. Let \( x(0) \) be the initial state vector containing the probabilities of a firm having each of four possible credit ratings at the initial point in time, \( t = 0 \) and consider the \( 4 \times 4 \) credit migration matrix in Figure 2. Each row of \( P \) represents the current credit
rating of the firm whereas each column represents the credit rating of the firm after $\Delta t$ [3]. Therefore, $p_{1,1} = 0.9$ represents the probability of a firm with an A credit rating maintaining an A credit rating after $\Delta t$, $p_{2,1} = 0.05$ represents the probability of a firm with a B credit rating receiving an A credit rating after $\Delta t$, and so forth. The probabilities in the last column of these matrices are especially important to lenders for the last column represents the probability of default given the obligors current rating [10]. For example, the probability of a firm with a B credit rating defaulting is $p_{2,4} = 0.05$.

\[
P = \begin{bmatrix}
0.9 & 0.07 & 0.03 & 0 \\
0.05 & 0.8 & 0.1 & 0.05 \\
0.01 & 0.05 & 0.84 & 0.1 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

Credit rating at $t = k + 1$

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.9</td>
<td>0.07</td>
<td>0.03</td>
<td>0</td>
</tr>
<tr>
<td>0.05</td>
<td>0.8</td>
<td>0.1</td>
<td>0.05</td>
</tr>
<tr>
<td>0.01</td>
<td>0.05</td>
<td>0.84</td>
<td>0.1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Credit rating at $t = k$

Figure 2: If a firm is at one of the given credit ratings at $t = k$, the corresponding row gives the probability of the firm transitioning to each credit rating at $t = k + 1$.

Also notice that once a firm is in default, it will not be able to return to a better credit rating. Mathematically this row is called an absorbing state [10].

To find the state vector $\mathbf{x}(1)$, which contains the number of the firms in the initial state having each credit rating after one fixed period of time $\Delta t$, the credit migration matrix $P$ is multiplied by $\mathbf{x}(0)$. For example, let $\mathbf{x}(0)$ be the initial state vector and $P$ be the transition matrix as previously defined. Figure 3 show the computation of $\mathbf{x}(1)$ through this method.
\[ x(0) = \begin{bmatrix} 40 & 60 & 20 & 20 \end{bmatrix} \]

\[ x(1) = x(0)P = \begin{bmatrix} 39.2 & 51.8 & 24 & 25 \end{bmatrix} \]

Figure 3: Computation of the next state vector.

By our assumption that the rows must sum to one, it follows that our dynamical system eventually reaches an equilibrium state [10]. Since we can find the \( k^{th} \) state vector by computing \( x(0)P^k \) [3], there exists some \( k \) at which \( P^k \approx P^{k-1} \). When the credit migration matrix reaches this equilibrium, the \( k^{th} \) state vector at this point is our final state vector [4, 12]. This final state vector shows how many firms from the initial state vector are in each credit rating after the dynamical system converges. This gives the lenders a long term outlook of the expected credit ratings of the firms in their portfolio based off the initial credit migration matrix.

2 Metrics for Credit Migration Matrices

When analyzing credit migration matrices, it is important to understand how matrices change over time and their statistical similarities [10]. Since every time a new state vector is computed, the current transition matrix is multiplied by the original matrix for the system, the transition matrix under consideration is different at every point in time until the dynamical system reaches equilibrium. It is convenient to have a compact, scalar representation of the properties of credit migration matrices to compare multiple matrices. This representation must also be comprehensive enough for meaningful comparison of matrices. This can be accomplished through metrics, or mobility indices [10]. The next few subsections
will define metrics, explain the criteria for mobility metrics, and introduce a currently used mobility metric.

2.1 Definition of Metrics

Metrics are continuous real functions over a transition matrix or a set of transition matrices. Since the main diagonal of the transition matrices represents no mobility between states, the off-diagonal entries, which determine the mobility of the matrices between states, are often analyzed mathematically to determine the mobility ranking of the credit migration matrix [15]. Metrics should be able to provide a scalar representation of important properties of credit migration matrices. A single number is helpful in that it can compactly characterize large amounts of information without users having manually analyze an entire matrix. Metrics also should provide an ordinal system for ranking credit migration matrices based on mobility [10]. There are several other criteria currently used in constructing mobility metrics.

2.2 Criteria for Mobility Metrics

Metrics for comparing credit migration matrices should be carefully analyzed. With the importance of these matrices in the field of risk management it should be verified that the metrics are providing useful, insightful information about these matrices. For example, when considering metrics which measure the level of mobility of a credit migration matrix, criteria are presented by Jafry and Schuermann in “Metrics for Comparing Credit Migration Matrices” [10]. There are three classes of criteria for mobility metrics: persistence criteria, convergence criteria, and temporal aggregation criteria. Persistence criteria are criteria
which imply that metrics must provide a simple, concise interpretation of the transition matrices. Convergence criteria state that metrics should establish a rank for the transition matrices consistent with the rate $P^k$ converges to the equilibrium transition matrix. Temporal aggregation criteria assure that differences in the length of $\Delta t$ do not influence the result of the metric [10]. We will focus on analyzing the persistence criteria in this work.

### 2.3 Average of the Singular Values of the Mobility Matrix

In Jafry and Schuermann’s 2003 work *Metrics for Comparing Credit Migration Matrices* a new metric defined to be the average of the singular values of the mobility matrix is presented. This metric is used to measure the amount of mobility expected given a single credit migration matrix. We will later use this metric to investigate the connection between financial outlook for the lender and the amount of mobility present in the matrix.

First, the mobility matrix is defined to be $P_M = P - I$ where $P$ is an $n \times n$ credit migration matrix and $I$ is the identity matrix of dimension $n$. This is performed to emphasize the off diagonal entries which represent migration to different credit ratings since credit migration matrices are diagonally dominant by assumption [10].

To compute the metric first we find $P_M'P_M$, the transpose of the mobility matrix multiplied by the mobility matrix. Then we find the singular values of this new matrix and compute the sum of the square roots of the $n$ singular values divided by $n$.

$$M_{SVD}(P) = \frac{\sum_{i=1}^{n} \sqrt{\lambda_i(P_M'P_M)}}{n}$$

This metric satisfies the persistence criteria in that it provides information about a credit migration matrix in a compact form [10]. We will compare the information this metric
provides with our newly proposed metric.

3 A New Metric for Credit Migration Matrices

While the previously mentioned metrics can be used to show how much movement between different credit ratings will happen over time, this information is not necessarily useful in showing the financial outlook of a lender’s current portfolio. As seen in Figure 4, two different matrices can have identical levels of mobility.

\[
\begin{bmatrix}
0.5 & 0.5 & 0 & 0 & 0 \\
0.5 & 0.5 & 0 & 0 & 0 \\
0 & 0 & 0.5 & 0 & 0 \\
0 & 0 & 0 & 0.5 & 0 \\
0 & 0 & 0 & 0 & 0.5 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
0.5 & 0.5 & 0 & 0 & 0 \\
0 & 0.5 & 0.5 & 0 & 0 \\
0 & 0 & 0 & 0.5 & 0 \\
0 & 0 & 0 & 0 & 0.5 \\
0 & 0 & 0 & 0 & 0.5 \\
\end{bmatrix}
\]

| Jafry’s Metric (Mobility): 0.588 | Jafry’s Metric (Mobility): 0.588 |

Figure 4: Matrices with same levels of mobility.

It could then be useful if there exists a metric which would give information about a credit migration matrix in terms of the matrix’s financial outlook for a lender’s portfolio. This metric should provide a simple, scalar interpretation of the matrix that allows users to distinguish between the desirability of multiple matrices. We propose a new metric which provides a scalar value which can be used as a long term prediction of the most common credit ratings among the firms in the portfolio. The following section defines our new metric.
3.0.1 Mathematical Definition

Suppose the dynamical system for the $n \times n$ transition matrix $P$ stabilizes as defined in Section 1.4.2 after transitioning through $k$ states. Let the state vector $x(k)$ be the final state vector containing the number of a total of $N$ firms in each of the $n$ credit ratings after the $k$ transitions. Let $s$ be the vector containing the possible credit ratings represented numerically as states $1, 2, 3, \cdots, n$ where 1 is the best credit rating and $n$ is the default rating. This vector will be used to weight the number of firms in each of the discrete credit ratings so these numbers must be sequential after $k$ transitions. The new metric $M$ finds the dot product between the final state vector and $s$ divided by the total number of firms in the state vector $N$ given the credit migration matrix $P$:

$$M(P) = \frac{x(k) \cdot s}{N}$$

This metric provides a scalar representation of the credit rating that the majority of firms considered will have after $k$ states. For the remainder of the text we will refer to the new metric as $M$.

For example, consider the credit migration matrix and initial state vector:

$$P = \begin{bmatrix} 0.9 & 0.1 & 0 \\ 0.1 & 0.8 & 0.1 \\ 0 & 0.2 & 0.8 \end{bmatrix}, \quad x(0) = \begin{bmatrix} 20 \\ 20 \\ 60 \end{bmatrix}$$
After computing the fifteenth state vector as defined in Section 1.4.2, the dynamical system has stabilized giving a final vector of

\[ x(15) = \begin{bmatrix} 40 & 40 & 20 \end{bmatrix} \]

Out of the original one hundred firms in the initial state vector, 40 are now in credit rating 1, 40 are in credit rating 2, and 20 are in credit rating 3, the default rating. Since there are 3 possible credit ratings we can let

\[ s = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \]

The numbers chosen to represent the credit ratings can be any sequence of successive numbers, but we chose these numbers for simplicity. In this arrangement a credit rating of 1 is more desirable than a credit rating of 3. Generalizing, the least desirable matrices will have a metric value of \( n \) given \( n \) states of the system and the most desirable metric value is 1.

Computing \( M \) we have:

\[
M(P) = \frac{x(15) \cdot s}{100} = \frac{[20 \ 20 \ 60] \begin{bmatrix} 0.9 & 0.1 & 0.1 \\ 0.1 & 0.8 & 0.1 \\ 0 & 0.2 & 0.8 \end{bmatrix}^{15} [1 \ 2 \ 3]}{100} = 1.8
\]

The value of \( M \) is 1.8 which means after reaching equilibrium a majority of the firms from the initial state vector will now have a 1 or 2 credit rating with more of these firms having a 2 rating since 1.8 is closer to 2 and the credit ratings are discrete. Translating back into letter based credit ratings, most firms will have an A or B credit rating which means
this matrix is fairly desirable.

4 Comparison of Metrics

While the metrics commonly used in industry describe movement of firms between different credit ratings, the newly proposed metric provides insight into how desirable the outlook of these matrices are to the lenders. A more desirable matrix will yield a better financial outcome. It is natural to investigate whether or not there is a connection between these two types of metrics. What does a matrix with a higher level of mobility infer about the financial outlook of a portfolio? Since mobility increases as more of the row-wise probability moves away from the diagonal of a migration matrix, there are two options for the financial outlook of the matrix. In the two extreme cases in Figure 5, matrices can have the same amount of mobility but on opposite sides of the main diagonal. If most of the probability is to the left of the diagonal, more firms will be transitioning to a better credit rating. However, if most of the probability is to the right of the diagonal, more firms will be transitioning to a worse credit rating.

\[
\begin{bmatrix}
0.5 & 0.5 & 0 & 0 & 0 \\
0.5 & 0.5 & 0 & 0 & 0 \\
0 & 0.5 & 0 & 0 & 0 \\
0 & 0 & 0.5 & 0.5 & 0 \\
0 & 0 & 0 & 0.5 & 0.5 \\
\end{bmatrix}
\begin{bmatrix}
0.5 & 0.5 & 0 & 0 & 0 \\
0 & 0.5 & 0.5 & 0 & 0 \\
0 & 0 & 0.5 & 0.5 & 0 \\
0 & 0 & 0 & 0.5 & 0.5 \\
0 & 0 & 0 & 0.5 & 0.5 \\
\end{bmatrix}
\]

New Metric: 1.5  
Jafry’s Metric (Mobility): 0.588

New Metric: 4.5  
Jafry’s Metric (Mobility): 0.588

Figure 5: Matrices with same levels of mobility but opposite desirability.
The two matrices have the exact same levels of mobility and have completely different desirability to lenders. In the first matrix the value of $M$ is 1.5 which is very desirable. The second matrix has a value for $M$ of 4.5 which is not desirable to lenders. However, both of the matrices have identical mobility with a mobility index of 0.588.

To perform further comparison between the two metrics we can then create series of matrices for which either mobility strictly increases or desirability strictly decreases. This will allow us to see the impact (if any) that mobility has on the financial desirability of the matrix and the impact which financial desirability has on mobility.

### 4.1 Method of Comparison

To compare the metric $M$ to Jafry’s (2003) metric, we used Visual Basic to create a spreadsheet to dynamically compute the values of our and Jafry’s metric for any entered $5 \times 5$ matrix and initial state vector. To perform the computations necessary for Jafry’s metric we also computed each matrix’s corresponding mobility matrix $P_M$ and then computed $P'_M P_M$. We then used Poptool’s Excel plugin in order to compute the eigenvalues of $P'_M P_M$ [10]. The following subsections explain the procedure used in this analysis.

#### 4.1.1 Construction of Matrices and Analysis

We first considered a sequence of five matrices in which the first matrix would have a very desirable outcome and the last matrix would have a very undesirable outcome. See matrices in Appendix 7.1. These matrices were constructed keeping the assumption that credit migration matrices are diagonally dominant. In each successive matrix probability was subtracted in the second row from the entry to the left of the diagonal and distributed to the
right of the diagonal to maintain the row-wise probability summing to one. In doing so, the newly proposed metric strictly increased from the first to last matrix (see Figure 6). Since the new metric is always on a scale between 0 and the maximum number of credit ratings being considered, the new metric is divided by the number of credit ratings so both metrics are always between 0 and 1. Doing so allows us to compare both metrics on the same scale which lets us more easily examine any possible linear connection between the two metrics.

![Comparison of Two Metrics](image)

Figure 6: The value of the the new metric and Jafry’s metrix while the new metric strictly increases.

As a key result of our work we found that while $M$ was strictly increasing, there is very little impact on the value of Jafry’s metric. This indicates that Jafry’s mobility metric does not provide information about the desirability of a given matrix. For matrices refer to Appendix 7.1.

Furthermore, we altered the mobility of credit migration matrices to see any possible
impact on desirability. We constructed matrices in which the first matrix was extremely immobile and the last matrix had relatively high mobility. It is complicated to construct a matrix which has perfect mobility while keeping the assumption that the matrices are diagonally dominant. It was also difficult to strongly impact the mobility of a matrix while only altering one row between matrices such as in the matrices with strictly increasing values of our metric. However, we feel further work can be done to investigate a possible connection between desirability and mobility. Therefore, we construct a sequence of five matrices in which mobility is strictly increasing while keeping this assumption and changing all rows of the matrix. Our first matrix has .95 of the row-wise probability on the diagonal while the remaining probability is split evenly on either side of the diagonal making this matrix highly immobile. In each matrix more probability is subtracted from the main diagonal and evenly distributed on either side of the main diagonal.
Figure 7: The value of the new metric and Jafry’s metric while the Jafry’s metric strictly increases.

While we see that the value of Jafry’s metric for these five matrices strictly increases, the value of the newly proposed metric increases between the first four matrices and decreases between the last two matrices. This could indicate that the financial desirability of a matrix is not affected by the mobility of the matrix. Therefore, $M$ is able to provide more information about a credit migration matrix than a mobility metric alone. See matrices in Appendix 7.2.

Through both of these comparisons and the example given in Figure 5 we conclude that the new metric $M$ lends more information about the desirability of credit migration matrices than a mobility based metric alone. This metric could be useful in comparison of different matrices to lenders performing risk management on their portfolios in conjunction with mobility indices. Future work can be done in trying to find a relation between $M$ and the
mobility of a matrix.

5 Analysis of The New Metric

Due to the credit migration matrices being diagonally dominant, it is useful to examine the impact the matrix entries on the main diagonal have on our metric. In other words it is useful to see how the metric behaves as the proportion of firms staying at each credit rating increases after a time step. For example, consider the matrix entry $p_{1,1}$ of the credit migration matrix $P$ from Figure 2. This entry represents firms which currently have an A credit rating remaining at an A credit rating after one transition. Since the rows of our transition matrices must add to 1, when we change the value of one of the diagonal entries, at least one of the entries in that row must also change to compensate for this change. If $p_{1,1}$ is 0 then it must be the case that $p_{1,2} = 1$ to ensure that the first row still sums to 1.

As a proof of concept we use a model with three possible credit ratings, A, B, and C. We will be considering the entries of our $3 \times 3$ transition matrix $P$, $p_{1,1}$, $p_{2,2}$, and $p_{3,3}$, which represent the probability of remaining at an A credit rating, a B credit rating, or a C credit rating respectively. We then further this procedure to an example which is more representative of industry.

When changing each diagonal entry and offsetting the probability to entries in the same row, the remaining rows of the matrix remain constant during analysis. Therefore, we also want to perform this analysis on matrices which differ in their desirability in these rows held constant.
5.1 Method of Analysis

We performed our analysis on different constructed matrices. In each matrix we need to analyze changes to each diagonal entry. For example in the $3 \times 3$ case we analyze changes in the entries $p_{1,1}$, $p_{2,2}$, and $p_{3,3}$. However, to consider different desirability in the rows remaining constant we consider three classes of matrices: those which are low risk, medium risk, and high risk for lenders. The low risk class of matrices are considered the most desirable matrices to lenders. Within these three categories we consider three more specific categories: best, middle, and worst. With three matrices for each of three categories, we need to analyze $n \times 9$ matrices total. Refer to our test matrices in the Appendix Sections 8.1 and 8.2.

5.1.1 Constructing Matrices 3 x 3

For each credit rating we constructed nine matrices with the best low risk matrix being the most desirable and the worst high risk matrix being the least desirable. The nine constructed matrices should have strictly increasing values of the new metric beginning at the best low risk matrix and progressing to worst high risk matrix since a higher value of this metric represents a worse credit rating in this model.

First, for each matrix we need to set up the row which will be changing during analysis to add to 1. We set the changing entry of each matrix to 0. When the changing entry is in the first or last row, the probability is always offset by setting the entry directly right or left of the changing entry to 1 respectively. In middle rows, the offsetting probability will be split with the entries directly left and right of the changing entry set to 0.5. For consistency, the nine matrices for each credit rating have the same entries in each row held constant in
each level of desirability. Refer to Appendix 8.1. for the $3 \times 3$ matrices used in the following analysis. For example consider the matrix:

$$\begin{bmatrix}
  x & y & 0 \\
  0.45 & 0.55 & 0 \\
  0 & 0.3 & 0.7
\end{bmatrix}$$

Initially we have $x = 0$ while $y = 1$. However if $x$ is increased then $y = 1 - x$. In middle rows (in this case only the second row) we would set the entries to the left and right of $x$ to $y/2$. We want to examine the behavior of the metric when changing the matrix in this manner until $x = 1$ and $y = 0$. For example, in Figure 8 the first row is changed so that $p_{1,1}$ increases while $p_{1,2}$ decreases. The second and third row are held constant.

$$\begin{bmatrix}
  0 & 1 & 0 \\
  0.45 & 0.55 & 0 \\
  0 & 0.3 & 0.7
\end{bmatrix} \quad \begin{bmatrix}
  0.25 & 0.75 & 0 \\
  0.45 & 0.55 & 0 \\
  0 & 0.3 & 0.7
\end{bmatrix} \quad \begin{bmatrix}
  0.50 & 0.50 & 0 \\
  0.45 & 0.55 & 0 \\
  0 & 0.3 & 0.7
\end{bmatrix} \quad \begin{bmatrix}
  0.75 & 0.25 & 0 \\
  0.45 & 0.55 & 0 \\
  0 & 0.3 & 0.7
\end{bmatrix} \quad \begin{bmatrix}
  1 & 0 & 0 \\
  0.45 & 0.55 & 0 \\
  0 & 0.3 & 0.7
\end{bmatrix}$$

Figure 8: Matrices examining changes in $p_{1,1}$ and $p_{1,2}$.

To do this we used Visual Basic to compute our metric as output for any valid transition matrix and initial state vector on a spreadsheet as input. See Appendix 10 for the VBA function written to compute the metric. For our system, we used an initial state vector containing 100 firms, 60 with an A credit rating and 20 with B and C credit ratings. Recall the values in the initial state vector are unimportant since initial conditions will not effect
the value of the metric. Therefore, the metric is computed for 100 matrices for each class, specific category, and credit rating. In our analysis we have three risk classes of matrices. We refined these classes so that matrices are ranked by risk within the classes 1, 2, or 3 with 1 denoting the least amount of risk to lenders.

5.1.2 Results for 3x3 Transition Matrices

As we incremented \( p_{1,1} \), the entry containing probability of remaining at an A credit rating in each matrix, the metric value of each matrix decreased. Since more firms with a credit rating are remaining at an A and a lower metric value is more desirable. In Figure 9 we see that the value of the metric improved for all nine matrices. Recall that a lower value of \( M \) is better. For matrices used in the analysis refer to Appendix 8.1.1.

![Figure 9: The value of the metric as \( p_{1,1} \) is changed.](image)

Next, we increment \( p_{2,2} \), the entry containing probability of remaining at a B credit rating, the metric value of low risk matrices (blue) increased, medium risk matrices (red)
remained fairly stable, and high risk matrices (green) decreased as shown in Figure 10. Since in low risk matrices less firms are transitioning from B to A and more are remaining at B, so the value of the metric should be higher and the matrices less desirable. On the other hand, in high risk matrices less firms are transitioning from B to C while more firms are remaining at B so the metric should decrease to a value more favorable to financial institutions. For matrices used in the analysis refer to Appendix 8.1.2.

Figure 10: The value of the metric as $p_{2,2}$ is changed.

Lastly, we increment $p_{3,3}$ the entry containing the probability of remaining at a C rating. The value of the metric increases for all nine matrices. The probability that a firm with a C credit rating will transition to a better credit rating decreases while the probability of the firms remaining at C increases so more firms will get stuck at the worst credit rating. This is shown in Figure 11. For matrices used in the analysis refer to Appendix 8.1.3.
We see that $M$ behaves as we would expect. The value of $M$ worsens as probability of transitioning to a worse credit rating increases. Thus, this supports that $M$ is a good measure of the desirability of credit migration matrix. See Appendix 10 for VBA code used to construct Figure 11.

5.1.3 Constructing Matrices 5 x 5

While the $3 \times 3$ matrices are useful to examine as a proof of concept, $5 \times 5$ transition matrices with states A, B, BB, C, and D are more representative of industry. The three major credit rating agencies in the U.S. each have A, B, C, and D ratings with A, B, and C having more specific ratings such as AA, BBB, etc. It is most representative to include the BB rating because a majority of the ratings issued fell into a B rating category of the respective rating agency. In 2011 the median rating issued by S & P in the U.S. was a BB and in Europe
a BBB [16]. Forty percent of all global ratings issued in 2013 by Fitch were in the BBB category [2]. A majority of credit ratings issued by Moody’s are in the B rating group [11].

The $5 \times 5$ matrices should be constructed by the same steps outlined in Section 5.1.1 used in constructing the matrices in the analysis with three possible credit ratings. However for the $5 \times 5$ matrices, we are able to tailor our matrices to be more realistic to industry to verify our metric provides meaningful information about more realistic credit migration matrices. Although we construct the $5 \times 5$ matrices using the previous steps, we want to add some additional assumptions to make these matrices more realistic. Besides these matrices being diagonally dominant, most of the remaining row-wise probability should be concentrated in the entries closer to the main diagonal. This shows that most of the firms which do not remain at the same credit rating either are upgraded or downgraded by only one rating or more rarely two ratings. We also apply the absorbing default state to the $5 \times 5$ matrices. Therefore the $p_{5,5}$ in each matrix will equal 1 [10]. Otherwise the process of analysis for the $5 \times 5$ matrices was identical to that of the $3 \times 3$ matrices.

5.1.4 Results for $5 \times 5$ Transition Matrices

In the $5 \times 5$ system our initial state vector contained 200 firms, 50 with an A credit rating, 80 with a B credit rating, 60 with a BB credit rating, 20 with a C credit rating, and 5 firms in default. Figure 12 shows the results when $p_{1,1}$ is incremented. The metric value of all nine matrices decreased showing that each matrix became more desirable as this entry is increased. Similar analysis of the next four credit ratings is in Appendix 9. We see that as the credit rating represented in the incremented entry worsens, more of the matrices yield a higher metric value at an increasing rate. For matrices used in analysis see Appendix 8.2.1.
Figure 12: The value of the metric as $p_{1,1}$ is changed

This further supports that $M$ behaves as expected and provides a measure of desirability of the matrices.

6 Conclusion and Future Work

While mobility indices currently used in industry are able to provide information on the expected amount of movement between credit ratings for firms in a portfolio, these numbers do not provide meaningful information on whether or not a credit migration matrix is desirable or not to lenders. The metric proposed in this paper, $M$, provides a means by which to compare matrices based on the desirability to lenders. Future study which could be explored further can be done in comparing this new metric to other mobility metrics to look for a further connection between mobility and desirability and ways in which this metric can be used in conjunction with mobility metrics to provide meaningful information about the
matrices.

References


Appendix

7 5x5 Matrices Used in Comparison of Metrics

7.1 Matrices with Strictly Increasing Value of our Metric

Ranked More Desirable to Less Desirable

(1)

\[
\begin{bmatrix}
0.95 & 0.05 & 0 & 0 & 0 \\
0.45 & 0.55 & 0 & 0 & 0 \\
0 & 0.15 & 0.7 & 0.15 & 0 \\
0 & 0 & 0.1 & 0.8 & 0.1 \\
0 & 0 & 0 & 0.05 & 0.95
\end{bmatrix}
\]

(2)

\[
\begin{bmatrix}
0.95 & 0.05 & 0 & 0 & 0 \\
0.35 & 0.55 & 0.05 & 0.05 & 0 \\
0 & 0.15 & 0.7 & 0.15 & 0 \\
0 & 0 & 0.1 & 0.8 & 0.1 \\
0 & 0 & 0 & 0.05 & 0.95
\end{bmatrix}
\]

(3)

\[
\begin{bmatrix}
0.95 & 0.05 & 0 & 0 & 0 \\
0.2 & 0.55 & 0.01 & 0.01 & 0 \\
0 & 0.15 & 0.7 & 0.15 & 0 \\
0 & 0 & 0.1 & 0.8 & 0.1 \\
0 & 0 & 0 & 0.05 & 0.95
\end{bmatrix}
\]

(4)

\[
\begin{bmatrix}
0.95 & 0.05 & 0 & 0 & 0 \\
0.15 & 0.55 & 0.15 & 0.15 & 0 \\
0 & 0.15 & 0.7 & 0.15 & 0 \\
0 & 0 & 0.1 & 0.8 & 0.1 \\
0 & 0 & 0 & 0.05 & 0.95
\end{bmatrix}
\]

(5)

\[
\begin{bmatrix}
0.95 & 0.05 & 0 & 0 & 0 \\
0.05 & 0.55 & 0.2 & 0.2 & 0 \\
0 & 0.15 & 0.7 & 0.15 & 0 \\
0 & 0 & 0.1 & 0.8 & 0.1 \\
0 & 0 & 0 & 0.05 & 0.95
\end{bmatrix}
\]
7.2 Matrices with Strictly Increasing Value of Jafry’s Mobility Metric

Ranked Immobile to Mobile

(1)
\[
\begin{bmatrix}
0.97 & 0.03 & 0 & 0 & 0 \\
0.025 & 0.95 & 0.025 & 0 & 0 \\
0 & 0.025 & 0.95 & 0.025 & 0 \\
0 & 0 & 0.025 & 0.95 & 0.025 \\
0 & 0 & 0 & 0.05 & 0.95
\end{bmatrix}
\]

(2)
\[
\begin{bmatrix}
0.9 & 0.05 & 0.05 & 0 & 0 \\
0.05 & 0.9 & 0.05 & 0 & 0 \\
0 & 0.05 & 0.9 & 0.05 & 0 \\
0 & 0 & 0.05 & 0.9 & 0.05 \\
0 & 0 & 0 & 0.1 & 0.9
\end{bmatrix}
\]

(3)
\[
\begin{bmatrix}
0.8 & 0.1 & 0.1 & 0 & 0 \\
0.01 & 0.8 & 0.1 & 0 & 0 \\
0 & 0.1 & 0.8 & 0.1 & 0 \\
0 & 0.1 & 0.7 & 0.1 & 0 \\
0 & 0 & 0.1 & 0.2 & 0.7
\end{bmatrix}
\]

(4)
\[
\begin{bmatrix}
0.7 & 0.1 & 0.1 & 0.1 & 0 \\
0.1 & 0.7 & 0.1 & 0.1 & 0 \\
0 & 0.025 & 0.95 & 0.025 & 0 \\
0 & 0 & 0.025 & 0.95 & 0.025 \\
0 & 0 & 0 & 0.05 & 0.95
\end{bmatrix}
\]

(5)
\[
\begin{bmatrix}
0.5 & 0.25 & 0.2 & 0.05 & 0 \\
0.1 & 0.5 & 0.25 & 0.1 & 0.05 \\
0 & 0.25 & 0.5 & 0.2 & 0.05 \\
0 & 0 & 0.25 & 0.5 & 0.25 \\
0 & 0 & 0.05 & 0.45 & 0.5
\end{bmatrix}
\]
8 Analysis of New Metric

In each matrix the entry $x$ is increased from 0 to 1 while $y$ is decreased from 1 to 0 to ensure the row-wise entries sum to 1.

8.1 3x3 Matrices Used in Analysis

#### 8.1.1 Transitions from A to A

<table>
<thead>
<tr>
<th></th>
<th>Best Case(1)</th>
<th>Average Case(2)</th>
<th>Worst Case(3)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Low</strong></td>
<td>$\begin{bmatrix} x &amp; y &amp; 0 \ 0.45 &amp; 0.55 &amp; 0 \ 0 &amp; 0.3 &amp; 0.7 \end{bmatrix}$</td>
<td>$\begin{bmatrix} x &amp; y &amp; 0 \ 0.35 &amp; 0.6 &amp; 0.05 \ 0 &amp; 0.25 &amp; 0.75 \end{bmatrix}$</td>
<td>$\begin{bmatrix} x &amp; y &amp; 0 \ 0.3 &amp; 0.6 &amp; 0.1 \ 0 &amp; 0.2 &amp; 0.8 \end{bmatrix}$</td>
</tr>
<tr>
<td><strong>Medium</strong></td>
<td>$\begin{bmatrix} x &amp; y &amp; 0 \ 0.2 &amp; 0.7 &amp; 0.1 \ 0 &amp; 0.15 &amp; 0.85 \end{bmatrix}$</td>
<td>$\begin{bmatrix} x &amp; y &amp; 0 \ 0.15 &amp; 0.7 &amp; 0.15 \ 0 &amp; 0.13 &amp; 0.87 \end{bmatrix}$</td>
<td>$\begin{bmatrix} x &amp; y &amp; 0 \ 0.1 &amp; 0.6 &amp; 0.3 \ 0 &amp; 0.1 &amp; 0.9 \end{bmatrix}$</td>
</tr>
<tr>
<td><strong>High</strong></td>
<td>$\begin{bmatrix} x &amp; y &amp; 0 \ 0 &amp; 0.55 &amp; 0.45 \ 0 &amp; 0.08 &amp; 0.92 \end{bmatrix}$</td>
<td>$\begin{bmatrix} x &amp; y &amp; 0 \ 0 &amp; 0.52 &amp; 0.48 \ 0 &amp; 0.05 &amp; 0.95 \end{bmatrix}$</td>
<td>$\begin{bmatrix} x &amp; y &amp; 0 \ 0.5 &amp; 0.5 &amp; 0 \ 0.02 &amp; 0.98 \end{bmatrix}$</td>
</tr>
</tbody>
</table>
### 8.1.2 Transitions from B to B

<table>
<thead>
<tr>
<th></th>
<th>Best Case(1)</th>
<th>Average Case(2)</th>
<th>Worst Case(3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Low</td>
<td>$\begin{bmatrix} 0.95 &amp; 0.05 &amp; 0 \ 0.5y &amp; x &amp; 0.5y \ 0 &amp; 0.3 &amp; 0.7 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0.9 &amp; 0.1 &amp; 0 \ 0.5y &amp; x &amp; 0.5y \ 0 &amp; 0.25 &amp; 0.75 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0.87 &amp; 0.1 &amp; 0.03 \ 0.5y &amp; x &amp; 0.5y \ 0 &amp; 0.2 &amp; 0.8 \end{bmatrix}$</td>
</tr>
<tr>
<td>Medium</td>
<td>$\begin{bmatrix} 0.85 &amp; 0.1 &amp; 0.05 \ 0.5y &amp; x &amp; 0.5y \ 0 &amp; 0.15 &amp; 0.85 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0.8 &amp; 0.1 &amp; 0.1 \ 0.5y &amp; x &amp; 0.5y \ 0 &amp; 0.13 &amp; 0.87 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0.75 &amp; 0.15 &amp; 0.1 \ 0.5y &amp; x &amp; 0.5y \ 0 &amp; 0.1 &amp; 0.9 \end{bmatrix}$</td>
</tr>
<tr>
<td>High</td>
<td>$\begin{bmatrix} 0.65 &amp; 0.25 &amp; 0.1 \ 0.5y &amp; x &amp; 0.5y \ 0 &amp; 0.08 &amp; 0.92 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0.6 &amp; 0.2 &amp; 0.2 \ 0.5y &amp; x &amp; 0.5y \ 0 &amp; 0.05 &amp; 0.95 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0.5 &amp; 0.25 &amp; 0.25 \ 0.5y &amp; x &amp; 0.5y \ 0 &amp; 0.02 &amp; 0.98 \end{bmatrix}$</td>
</tr>
</tbody>
</table>
8.1.3 Transitions from C to C

<table>
<thead>
<tr>
<th></th>
<th>Best Case(1)</th>
<th>Average Case(2)</th>
<th>Worst Case(3)</th>
</tr>
</thead>
</table>
| Low      | \[
\begin{bmatrix}
0.95 & 0.05 & 0 \\
0.45 & 0.55 & 0 \\
0 & y & x
\end{bmatrix}
\] | \[
\begin{bmatrix}
0.9 & 0.1 & 0 \\
0.35 & 0.6 & 0.05 \\
0 & y & x
\end{bmatrix}
\] | \[
\begin{bmatrix}
0.87 & 0.1 & 0.03 \\
0.3 & 0.6 & 0.1 \\
0 & y & x
\end{bmatrix}
\] |
| Medium   | \[
\begin{bmatrix}
0.85 & 0.1 & 0.05 \\
0.2 & 0.7 & 0.1 \\
0 & y & x
\end{bmatrix}
\] | \[
\begin{bmatrix}
0.8 & 0.1 & 0.1 \\
0.15 & 0.7 & 0.15 \\
0 & y & x
\end{bmatrix}
\] | \[
\begin{bmatrix}
0.75 & 0.15 & 0.1 \\
0.1 & 0.6 & 0.3 \\
0 & y & x
\end{bmatrix}
\] |
| High     | \[
\begin{bmatrix}
0.65 & 0.25 & 0.1 \\
0 & 0.55 & 0.45 \\
0 & y & x
\end{bmatrix}
\] | \[
\begin{bmatrix}
0.6 & 0.2 & 0.2 \\
0 & 0.52 & 0.48 \\
0 & y & x
\end{bmatrix}
\] | \[
\begin{bmatrix}
0.5 & 0.25 & 0.25 \\
0 & 0.5 & 0.5 \\
0 & y & x
\end{bmatrix}
\] |
### 8.2 5x5 Matrices Used in Analysis

#### 8.2.1 Transitions from A to A

<table>
<thead>
<tr>
<th>Low</th>
<th>Average Case(2)</th>
<th>Worst Case(3)</th>
</tr>
</thead>
<tbody>
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<td><img src="image1" alt="Matrix" /></td>
<td><img src="image2" alt="Matrix" /></td>
<td><img src="image3" alt="Matrix" /></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Medium</th>
<th>Average Case(2)</th>
<th>Worst Case(3)</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image4" alt="Matrix" /></td>
<td><img src="image5" alt="Matrix" /></td>
<td><img src="image6" alt="Matrix" /></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>High</th>
<th>Average Case(2)</th>
<th>Worst Case(3)</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image7" alt="Matrix" /></td>
<td><img src="image8" alt="Matrix" /></td>
<td><img src="image9" alt="Matrix" /></td>
</tr>
</tbody>
</table>
### 8.2.2 Transitions from B to B

<table>
<thead>
<tr>
<th></th>
<th>Best Case(1)</th>
<th>Average Case(2)</th>
<th>Worst Case(3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Low</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.97 0.03 0 0 0</td>
<td>0.95 0.05 0 0 0</td>
<td>0.9 0.07 0.03 0 0</td>
<td></td>
</tr>
<tr>
<td>0.5y x 0.5y 0 0</td>
<td>0.5y x 0.5y 0 0</td>
<td>0.5y x 0.5y 0 0</td>
<td></td>
</tr>
<tr>
<td>0.05 0.25 0.6 0.1 0</td>
<td>0 0.3 0.6 0.1 0</td>
<td>0 0.3 0.57 0.13 0</td>
<td></td>
</tr>
<tr>
<td>0.05 0.1 0.3 0.5 0.05</td>
<td>0 0.15 0.25 0.5 0.1</td>
<td>0 0.1 0.3 0.5 0.1</td>
<td></td>
</tr>
<tr>
<td>0 0 0 0 1</td>
<td>0 0 0 0 1</td>
<td>0 0 0 0 1</td>
<td></td>
</tr>
<tr>
<td>Medium</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.9 0.05 0.05 0 0</td>
<td>0.85 0.1 0.05 0 0</td>
<td>0.8 0.15 0.05 0 0</td>
<td></td>
</tr>
<tr>
<td>0.5y x 0.5y 0 0</td>
<td>0.5y x 0.5y 0 0</td>
<td>0.5y x 0.5y 0 0</td>
<td></td>
</tr>
<tr>
<td>0 0.25 0.52 0.2 0.03</td>
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<td></td>
</tr>
<tr>
<td>0 0.05 0.3 0.55 0.1</td>
<td>0 0 0.25 0.6 0.15</td>
<td>0 0 0.2 0.65 0.15</td>
<td></td>
</tr>
<tr>
<td>0 0 0 0 1</td>
<td>0 0 0 0 1</td>
<td>0 0 0 0 1</td>
<td></td>
</tr>
<tr>
<td>High</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.65 0.2 0.15 0 0</td>
<td>0.55 0.25 0.1 0.1 0</td>
<td>0.5 0.2 0.15 0.1 0.05</td>
<td></td>
</tr>
<tr>
<td>0.5y x 0.5y 0 0</td>
<td>0.5y x 0.5y 0 0</td>
<td>0.5y x 0.5y 0 0</td>
<td></td>
</tr>
<tr>
<td>0 0.1 0.6 0.2 0.1</td>
<td>0 0.05 0.5 0.25 0.2</td>
<td>0 0 0.55 0.25 0.2</td>
<td></td>
</tr>
<tr>
<td>0 0 0.15 0.7 0.15</td>
<td>0 0 0.1 0.7 0.2</td>
<td>0 0 0.05 0.65 0.3</td>
<td></td>
</tr>
<tr>
<td>0 0 0 0 1</td>
<td>0 0 0 0 1</td>
<td>0 0 0 0 1</td>
<td></td>
</tr>
</tbody>
</table>
### 8.2.3 Transitions from BB to BB

<table>
<thead>
<tr>
<th></th>
<th>Best Case(1)</th>
<th>Average Case(2)</th>
<th>Worst Case(3)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Low</strong></td>
<td><img src="matrix1.png" alt="Matrix" /></td>
<td><img src="matrix2.png" alt="Matrix" /></td>
<td><img src="matrix3.png" alt="Matrix" /></td>
</tr>
<tr>
<td><strong>Medium</strong></td>
<td><img src="matrix4.png" alt="Matrix" /></td>
<td><img src="matrix5.png" alt="Matrix" /></td>
<td><img src="matrix6.png" alt="Matrix" /></td>
</tr>
<tr>
<td><strong>High</strong></td>
<td><img src="matrix7.png" alt="Matrix" /></td>
<td><img src="matrix8.png" alt="Matrix" /></td>
<td><img src="matrix9.png" alt="Matrix" /></td>
</tr>
</tbody>
</table>
### 8.2.4 Transitions from C to C

<table>
<thead>
<tr>
<th></th>
<th>Best Case(1)</th>
<th>Average Case(2)</th>
<th>Worst Case(3)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Low</strong></td>
<td><img src="image1" alt="Matrix" /></td>
<td><img src="image2" alt="Matrix" /></td>
<td><img src="image3" alt="Matrix" /></td>
</tr>
<tr>
<td><strong>Medium</strong></td>
<td><img src="image4" alt="Matrix" /></td>
<td><img src="image5" alt="Matrix" /></td>
<td><img src="image6" alt="Matrix" /></td>
</tr>
<tr>
<td><strong>High</strong></td>
<td><img src="image7" alt="Matrix" /></td>
<td><img src="image8" alt="Matrix" /></td>
<td><img src="image9" alt="Matrix" /></td>
</tr>
</tbody>
</table>

Note: The matrices represent transition probabilities in different cases.
### 8.2.5 Transitions from D to D

<table>
<thead>
<tr>
<th></th>
<th>Best Case(1)</th>
<th>Average Case(2)</th>
<th>Worst Case(3)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Low</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.97 0.03 0 0 0</td>
<td>0.95 0.05 0 0 0</td>
<td>0.9 0.07 0.03 0 0</td>
</tr>
<tr>
<td></td>
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<td>0.25 0.7 0.05 0 0</td>
<td>0.25 0.65 0.1 0 0</td>
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<td>0 0 0 y x</td>
<td>0 0 0 y x</td>
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<tr>
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</tr>
<tr>
<td></td>
<td>0 0 0 y x</td>
<td>0 0 0 y x</td>
<td>0 0 0 y x</td>
</tr>
<tr>
<td><strong>High</strong></td>
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<tr>
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<td>0.55 0.25 0.1 0.1</td>
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</tr>
<tr>
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<td>0.05 0.52 0.2 0.2 0.03</td>
<td>0 0.5 0.25 0.1 0.15</td>
</tr>
<tr>
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</tr>
<tr>
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9 Analysis of 5x5 Credit Migration Matrices

Figure 13: The value of the metric as $p_{2,2}$ changes. See matrices in Appendix 8.2.2.

Figure 14: The value of the metric as $p_{3,3}$ changes. See matrices in Appendix 8.2.3.
Figure 15: The value of the metric as $p_{4,4}$ changes. See matrices in Appendix 8.2.4.

Figure 16: The value of the metric as $p_{5,5}$ changes. See matrices in Appendix 8.2.5.
Code to Compute Final State Vector and New Metric

Public Function finalVector(tmatrix As Range, state0 As Range) As Variant()
Dim transMatrix() As Variant, state_0() As Variant, prevState2() As Variant
Dim nextState2() As Variant, nextTransMatrix() As Variant

' range to array
transMatrix = tmatrix
state_0 = state0

'set state_02 as the previous state, copy values
ReDim prevState2(1 To 1, 1 To UBound(state_0, 2))
Dim i As Integer
For i = 1 To UBound(state_0, 2)
    prevState2(1, i) = state0(1, i)
Next i

' find the next state
ReDim nextState2(1 To 1, 1 To UBound(state_0, 2))
nextState2 = Application.WorksheetFunction.MMult(state_0, transMatrix)

' counter for loop
Dim k As Integer
For k = 1 To 1000

' find the next transition matrix
    If k = 1 Then
        nextTransMatrix = Application.WorksheetFunction.MMult(transMatrix, transMatrix)
    Else
        nextTransMatrix = Application.WorksheetFunction.MMult(transMatrix, nextTransMatrix)
    End If

' set prevState to current next state, copy entries
ReDim prevState2(1 To 1, 1 To UBound(state_0, 2))
Dim m As Integer
For m = 1 To UBound(prevState2, 2)
    prevState2(1, m) = nextState2(m)
Next m
'find nextState
ReDim nextState2(1 To 1, 1 To UBound(state_0, 2))
nextState2 = Application.WorksheetFunction.MMult(prevState2,
nextTransMatrix)

Next k

'return the final state vector
finalVector = nextState2

End Function

Public Function finalmetric(finalstate As Range) As Double
Dim creditRatings() As Integer, finState() As Variant

'Range to array
finState = finalstate

'Fill creditRatings with the possible credit ratings
ReDim creditRatings(1 To 1, 1 To UBound(finState, 2))
Dim k As Integer
For k = 1 To UBound(finState, 2)
    creditRatings(1, k) = k
Next k

'Returns metric
finalmetric =
Application.WorksheetFunction.SumProduct(finState, creditRatings) / 
Application.WorksheetFunction.Sum(finState)

End Function
Sub graph()
Dim metricGood() As Variant, xaxis() As Variant, metricOkay() As Variant
Dim metricBad() As Variant, k As Integer, delta As Double
delta = 0.01
ReDim metricGood(0 To 100)
ReDim metricOkay(0 To 100)
ReDim metricBad(0 To 100)
ReDim xaxis(0 To 100)
Dim aGood As Chart

For k = 0 To 100
    xaxis(k) = delta * k
    'Good matrix
    Cells(12, 12) = delta * k
    Cells(12, 10) = 1 - (delta * k)
    metricGood(k) = finalmetric(Range("B41:D41"))

    'Okay matrix
    Cells(17, 12) = delta * k
    Cells(17, 11) = (1 - delta * k) / 2
    Cells(17, 10) = (1 - delta * k) / 2
    metricOkay(k) = finalmetric(Range("B42:D42"))

    'Bad matrix
    Cells(37, 7) = delta * k
    Cells(37, 8) = 1 - (delta * k)
    metricBad(k) = finalmetric(Range("B43:D43"))

    If k Mod 10 = 0 Then
        Worksheets("Sheet3").Cells(4 + (k / 10), 14) = metricBad(k)
    End If
Next k

Set aGood = ActiveWorkbook.Charts.Add
Set aGood = aGood.Location(Where:=xlLocationAsObject, Name:="Sheet2")
With aGood
    .ChartType = xlXYScatterSmooth
End With

Dim g As Series
Set g = aGood.SeriesCollection.NewSeries
With g
    .Values = metricGood
.XValues = xaxis
.MarkerStyle = xlMarkerStyleNone
.Name = "Good"
End With

Dim o As Series
Set o = aGood.SeriesCollection.NewSeries
With o
 .Values = metricOkay
 .XValues = xaxis
 .MarkerStyle = xlMarkerStyleNone
 .Name = "Okay"
End With

Dim b As Series
Set b = aGood.SeriesCollection.NewSeries
With b
 .Values = metricBad
 .XValues = xaxis
 .MarkerStyle = xlMarkerStyleNone
 .Name = "Bad"
End With

With aGood.Axes(xlCategory)
 .MajorUnit = 0.5
 .MaximumScale = [1]
End With

With aGood.Axes(xlValue)
 .MajorUnit = 1
 .HasMajorGridlines = False
 .MinimumScale = [1]
 .MaximumScale = [3]
End With

With aGood
 .Axes(xlCategory, xlPrimary).HasTitle = True
 .Axes(xlCategory, xlPrimary).AxisTitle.Characters.Text = "Probability of Transition from C to C"
 .Axes(xlValue, xlPrimary).HasTitle = True
 .HasLegend = True
End With

End Sub